Small noise asymptotic expansion for a infinite dimensional stochastic reaction-diffusion forced Van der Pol equation

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Abstract- Starting from the classical *Van der Pol equation*, after suitable changes of variables, we derive a reaction-diffusion type forced Van der Pol equation with values in a suitable infinite dimensional Hilbert space. In particular we will perturb previous equation with a small additive Brownian noise. After some preliminary results concerning the non trivial existence and uniqueness of a solution, due to the presence of a non-Lipschitz non-linearity, we provide a rigorous asymptotic expansions in term of the small parameter ε of the related solution up to order 3. We will then explicitly write the first three order of the rigorous expansion and provide as well an upper bound for the remainder.

Key–Words- Asymptotic expansions, Stochastic reaction-diffusion systems, Van der Pol oscillator.

1 Introduction

A dynamical system exhibiting a stable periodic orbit is called an oscillator. Oscillators play an important role in physic due to the wide variety of phenomena they can model, from the harmonic oscillator to its quantum counterpart. Within this framework particular attention has been devoted to the study of the so called *Van der Pol oscillator*, a particular model studied by *Balthazar van der Pol*, see [24, 25, 26, 27], during his studies on electronic circuits when he first noticed stable oscillations.

The Van der Pol model is characterized by a non-linear damping of strength parameter μ, and it is governed by the second order differential equation

$$\ddot{x} - \mu(1-x^2)\dot{x} + x = 0\,. \tag{1}$$

In what follows we will focus our attention on a generalization of the previous model, namely the *forced Van der Pol oscillator* which is characterized by

$$\ddot{x} - \mu(1-x^2)\dot{x} + x = A\sin(\omega t)\,, \tag{2}$$

where A is the amplitude of the wave function and ω is its angular velocity. It is worth to mention that the dynamic of the model is strictly related to the value of the damping parameter μ.

In particular we will take into consideration the *large damping* case, namely when $\mu >> 1$, which is usually referred as the *relaxation oscillations régime*, see, e.g. [24]. Following [16] we can reduce eq. (2) to a system of two equations. In particular rescaling the time variable $t = \frac{t}{\mu}$ and defining $y := \frac{\dot{x}}{\mu^2} + \frac{x^3}{3} - x$ we have that eq. (2) reads

$$\begin{cases} \frac{1}{\mu^2}\dot{x} = y - \frac{x^3}{3} + x\,, \\ \dot{y} = -x + A\sin(\omega\mu t)\,, \end{cases} , \tag{3}$$

see, e.g. [16].

We would like to underline that eq. (1), up to suitable modifications, is intensively used to describe spike generation processes in giant squid axons, as well as in *FitzHugh-Nagumo* type systems, see, e.g. [13, 21], in the *Burridge–Knopoff model*, which is used to describe earthquake characterized by viscous friction, see, e.g. [7], and to describe certain energy market frameworks as pointed out in [19].

The work is structured as follows: in Sec. 2 we introduce the infinite dimensional setting and we will prove some fundamental properties of the infinite dimensional stochastic Van der Pol equation, while in Sec. 3 we will state the results which show the validity of the asymptotic expansion in powers of a small parameter ε, together with the existence and uniqueness for the solution of the related equations, eventually in Sec. 4 we will state the main results.

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2 The infinite-dimensional setting

We will consider the infinite dimensional stochastic reaction-diffusion version of the forced Van der Pol equation (3) taking values in a suitable Hilbert space H endowed with boundary Neumann condition. In particular, according to the approach introduced in [11], we will consider a stochastic partial differential equation (SPDE), with a smooth nonlinearity, driven by a cylindrical Wiener process taking values in an infinite dimensional Hilbert space. Eventually we will consider a small parameter ε in front of the Wiener process.

Setting $\delta := \frac{1}{\mu}$, taking the time, resp. space, variable $t \in [0,\infty)$, resp. $\xi \in [0,1]$, while v and w are real valued random variables, γ is strictly positive phenomenological constants, c and b are strictly positive smooth functions on $[0,1]$ and $0 < \xi_1 < 1$ is a characteristic value of the system such that, denoting by $\bar{b} := \min_\xi b(\xi)$, it holds that

$$3\bar{b} - (\xi_1^2 - \xi_1 + 1) \geq 0\,, \tag{4}$$

and eq. (3) reads as follows

$$\begin{cases} \partial_t v(t,\xi) = \partial_\xi (c(\xi)\partial_\xi v(t,\xi)) \\ \qquad - v(t,\xi)(v(t,\xi)-1)(v(t,\xi)-\xi_1) + \\ \qquad - b(\xi)v(t,\xi) - w(t,\xi) + \varepsilon \partial_t \beta_1(t,\xi)\,, \\ \partial_t w(t,\xi) = \delta(\gamma v(t,\xi) - A\sin(\omega t)) + \varepsilon \partial_t \beta_2(t,\xi)\,, \\ \partial_\xi v(t,0) = \partial_\xi v(t,1) = 0\,, \quad t \in [0,\infty) \\ v(0,\xi) = v_0(\xi), \quad w(0,\xi) = w_0(\xi)\,, \quad \xi \in [0,1]. \end{cases}, \tag{5}$$

which has been studied in [1, 6].

We further prescribe Neumann boundary conditions and we assume the initial values v_0 and w_0 to be in $C([0,1])$. Moreover we consider a small parameter $\varepsilon > 0$ in front of the noise, where we have denoted by β_1 and β_2 two independent Q_i-Brownian motions, $i = 1,2$, Q_i being positive trace class commuting operators, we refer the reader to [11] for an detailed treatment of random perturbations taking values in infinite dimensional Hilbert spaces.

In order to rewrite (5) in a more compact form as an infinite dimensional stochastic evolution equation, let us start considering a vector

$$u = \begin{pmatrix} v \\ w \end{pmatrix} \in H\,,$$

where H is the separable Hilbert space

$$H := L^2([0,1]) \times L^2([0,1])\,,$$

endowed with the inner product

$$\langle (v_1,w_1),(v_2,w_2)\rangle_H = \gamma \langle v_1, v_2\rangle_{L^2} + \langle w_1, w_2\rangle_{L^2}\,, \tag{6}$$

where $\langle \cdot,\cdot \rangle$ denotes the usual scalar product in $L^2([0,1])$, and the corresponding norm will be indicated by $|\cdot|$, then we define the operator $A : D(A) \subset H \to H$ as follows

$$A = \begin{pmatrix} A_0 - b & -I \\ \delta\gamma I & 0 \end{pmatrix}, \quad A_0 = \partial_\xi \left(c(\xi)\partial_\xi \right)\,,$$

with domain given by

$$D(A) := D(A_0) \times L^2([0,1])\,,$$
$$D(A_0) := \{u \in H^2([0,1]) : \partial_\xi v(t,0) = \partial_\xi v(t,1) = 0\},$$

and the non-linear operator

$$F : D(F) := L^6([0,1]) \times L^2([0,1]) \to H\,,$$

given by

$$F\left(\begin{pmatrix} v \\ w \end{pmatrix} \right) = \begin{pmatrix} -v(v-1)(v-\xi_1) \\ \delta\gamma A \sin(\omega t) \end{pmatrix}.$$

From the separability of the Hilbert space H we have that it exists an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ made of eigenvalues of A_0 such that the following bound holds

$$\exists M > 0, |e_k(\xi)| \leq M, \xi \in [0,1], k \in \mathbb{N}\,.$$

Let us thus consider the filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$, such that the two independent Wiener processes β_1 and β_2 are adapted to the filtration $\mathscr{F}_t$, $\forall\, t \geq 0$, and

$$\beta_i \in C\left([0,T]; L^2(\Omega, L^2(0,1))\right), i = 1,2,$$

with $\mathscr{L}(\beta_i(t)) \sim \mathscr{N}\left(0, t\sqrt{Q_i}\right)$, $i = 1,2$, with Q_i a linear operator on $L^2([0,1])$. We can assume that the operators Q_i are of trace class and that they diagonalize on the same basis $\{e_k\}_{k\in\mathbb{N}}$ of the Hilbert space H, namely $Q_i e_k = \lambda_k^i e_k$, $i = 1,2$. We also assume that $\sum_{i=1}^2 \sum_{k=1}^\infty \lambda_k^i < \infty$, and we denote by $W(t)$ a cylindrical Wiener process on H and by Q the operator

$$Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}\,.$$

Exploiting previously introduced notations, eq. (5) can be rewritten as follows

$$\begin{cases} du(t) = [Au(t) + F(u(t))]dt + \varepsilon\sqrt{Q}dW(t), \\ u(0) = u^0 \in H\,, t \in [0,+\infty)\,, \end{cases} ; \tag{7}$$

the main problem when dealing with eq. (7) is the non-linear term F, since standard *existence and uniqueness* results for equations of the type (7) are given for a Lipschitz non-linearity F, which is not our case since we have to deal with a polynomial of order 3.

Such a model is the basis for a pletora of applications spanning from statistical to quantum mechanics, from neurobiology to finance, etc. The aim of the present work is to provide an asymptotic expansions of the solution to all orders in the perturbation parameter ε, with explicit expressions both for the expansion coefficients and the remainder. Latter type of results are particularly relevant from a theoretical point of view as well as from the numerical side since they allow for concrete and accurate numerical simulations.

3 Assumptions and Basic results

Let us first consider the following deterministic problem

$$\begin{cases} du_0(t) = [Au_0(t) + F(u_0(t))]dt, t \in [0, +\infty) \\ du_0(0) = u^0 \in D(F) \end{cases} \tag{8}$$

whit notations as in Sec. 1. We want to study a stochastic perturbation, characterized by a small intensity parameter ε, of the eq. (8) writing its (unique) solution as an expansion in powers of $\varepsilon > 0$ as ε goes to zero. In particular, taking $t \in [0, +\infty)$, we are concerned with the following stochastic Cauchy problem on the Hilbert space H

$$\begin{cases} du(t) = [Au(t) + F(u(t))]dt + \varepsilon\sqrt{Q}dW(t), \\ u(0) = u^0 \in H, \end{cases} \tag{9}$$

for which we show that a solution u can be written in power series as

$$u(t) = u_0(t) + \varepsilon u_1(t) + \cdots + \varepsilon^n u_n(t) + R_n(t, \varepsilon), \tag{10}$$

where $u_0(t)$ solves the deterministic problem (8), $u_1(t)$ solves

$$\begin{cases} du_1(t) = [Au_1(t) + \nabla F(u_0(t))[u_1(t)]]dt + \sqrt{Q}dW(t), \\ u_1(0) = 0, \quad t \in [0, +\infty), \end{cases} \tag{11}$$

where we have denoted by

$$\nabla F(x)[h] = \lim_{s \to 0} \frac{F(x + sh) - F(x)}{s},$$

the Gâteaux derivative in the h direction, see, e.g. [1]. Eventually the $k-$th term $u_k(t)$ solves

$$\begin{cases} du_k(t) = [Au_k(t) + \nabla F(u_0(t))[u_k(t)]]\, dt + \Phi_k(t)dt \\ u_k(0) = 0 \end{cases} ; \tag{12}$$

where $\Phi_k(t)$ depends on $u_1(t), \ldots, u_{k-1}(t)$ and the derivatives of F up to order k. In order to guarantee existence and uniqueness results for eq. (9) and the validity of the power expansion (10), we need some regularity properties for the involved operators. In particular we have the following result.

Proposition 1. *Under the setting introduced in Sec. 1, we have*

(i) *$A : D(A) \subset H \to H$ generates an analytic semigroup e^{tA} of strict negative type such that*

$$\left\|e^{tA}\right\|_{\mathscr{L}(H)} \le e^{-\omega t}, \quad t \ge 0 \tag{13}$$

with $\omega > 0$;

(ii) *The map $F : D(F) \to H$ is continuous, Fréchet differentiable and there exist positive real numbers η and κ such that for any $u_1, u_2 \in D(F)$*

$$\begin{aligned} &\langle F(u_1) - F(u_2) - \eta(u_1 - u_2), u_1 - u_2\rangle < 0, \\ &|F(u)|_H \le \kappa(1 + |u|_H^3), \quad u \in D(F), \end{aligned} ; \tag{14}$$

(iii) *the term $A + F$ is $m-$dissipative, namely we have $\omega - \eta > 0$;*

(iv) *F is Fréchet differentiable and for any $i = 0, 1, 2, \ldots$*

$$\|\nabla^{(i)}F(u)\|_{\mathscr{L}(H,\mathscr{L}(H,H))} \le \kappa_i(1 + |u|_H^{3-i}),; \tag{15}$$

Proof. **(i)** Let us set

$$\bar{c} = \min_{\xi} c(\xi) > 0,$$

then we have that

$$\begin{aligned} &\int_0^1 \partial_\xi(c(\xi)\partial_\xi u(\xi))u(\xi)d\xi = \\ &= c(\xi)u(\xi)\partial_\xi u(\xi)\big|_0^1 - \le -\int_0^1 c(\xi)\left(\partial_\xi u(\xi)\right)^2 d\xi \\ &\le \bar{c}|Du|_{L^2}^2 \le 0. \end{aligned}$$

Setting then $\bar{b} = \min_\xi b(\xi) > 0$ and taking $u \in H$, it follows that

$$\begin{aligned} \langle Au, u\rangle &= \gamma\langle A_0 u_1, u_1\rangle - \gamma\bar{b}\langle u_1, u_1\rangle \\ &\quad - \gamma\langle u_1, u_2\rangle + \gamma\delta\langle u_1, u_2\rangle \le -\bar{b}|u_1|_{L^2}^2 \le 0, \end{aligned}$$

which proves eq. (13);

(ii) $D(F) = L^6([0,1]) \times L^2([0,1])$, and being F is a polynomial of degree 3, both continuity and the Féchet differentiability are straightforward.

Taking then $\eta := \frac{1}{3}(\xi_1^2 - \xi_1 + 1)$ we have that

$$\begin{aligned} &\langle F(u_1) - F(u_2) - \eta(u_1 - u_2), u_1 - u_2\rangle_H = \\ &-\gamma\langle v_1(v_1 - 1)(v_1 - \xi_1) \\ &- v_2(v_2 - 1)(v_2 - \xi_1), v_1 - v_2\rangle_{L^2} \\ &- \gamma\eta\langle v_1 - v_2, v_1 - v_2\rangle_{L^2} \\ &- \gamma\eta\langle w_1 - w_2, w_1 - w_2\rangle_{L^2}, \end{aligned}$$

denoting now $p(x) := x(x-1)(x-\xi_1)$, we have

$$p(v_1) - p(v_2) \le \sup_\zeta p'(\zeta)(v_1 - v_2).$$

Therefore we have that

$$\begin{aligned} &\langle F(u_1) - F(u_2) - \eta(u_1 - u_2), u_1 - u_2\rangle_H \le \\ &\gamma\left[\sup_\zeta p'(\zeta) - \eta\right]|v_1 - v_2|_{L^2}^2 - \gamma\eta|w_1 - w_2|_{L^2}^2, \end{aligned}$$

it can now be easily seen that $\sup_\zeta p'(\zeta) = \frac{1}{3}(\xi_1^2 - \xi_1 + 1) = \eta$, so that the first term vanish and the claim follows. The second estimate in eq. (14) immediately follows from the fact that F is a polynomial of degree 3;

(iii) it follows from assumption (4) so that we have $\bar{b} - \eta > 0$ and $A + F$ is *m-dissipative*;

(iv) From the particular form of F it immediately follows

$$\nabla^{(1)}F\left(\begin{pmatrix} v \\ w \end{pmatrix}\right) = \begin{pmatrix} -3v^2 + 2v(1+\xi_1) - \xi_1 \\ 0 \end{pmatrix},$$
$$\nabla^{(2)}F\left(\begin{pmatrix} v \\ w \end{pmatrix}\right) = \begin{pmatrix} -6v + 2(1+\xi_1) \\ 0 \end{pmatrix},$$
$$\nabla^{(3)}F\left(\begin{pmatrix} v \\ w \end{pmatrix}\right) = \begin{pmatrix} -6 \\ 0 \end{pmatrix},$$
$$\nabla^{(j)}F\left(\begin{pmatrix} v \\ w \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, j \geq 4,$$

and estimate (15) follows since the $k-$th derivative is a polynomial of order $3-k$.

□

3.1 Preliminary results

Definition 2. *Let $u^0 \in D(F)$; we say that $u_0 = u_0(t)$ is a mild solution of* (8) *if it is $t-$continuous and it satisfies*

$$u_0(t) = e^{tA}u^0 + \int_0^t e^{(t-s)A}F(u_0(s))ds \ , \ \forall t \in [0,+\infty) \ . \qquad (16)$$

A predictable H-valued process $u(t)$, $t \in [0,+\infty)$ is said to be a mild solution to (7) *iff*

$$u(t) = e^{tA}u^0 + \int_0^t e^{(t-s)A}F(u(s))ds + \varepsilon \int_0^t e^{(t-s)A}\sqrt{Q}dW(s) \ , \ \mathbb{P}-a.s.$$

Remark 3. *We would like to recall that the last term in the previous equation, namely*

$$W_A(t) := \int_0^t e^{(t-s)A}\sqrt{Q}dW(s) \, ,$$

is usually called stochastic convolution, and it is a well defined (by Hypothesis) mean square continuous $\mathscr{F}_t-$adapted Gaussian process, see, e.g., [DPZ].

The following fundamental result ensures the existence and the uniqueness of the solution to eq. (7)

Proposition 4. *The Stochastic Convolution $W_A(t)$ is $\mathbb{P}-$almost surely continuous for $t \in [0,+\infty)$ and it verifies*

$$\mathbb{E}\left[\sup_{t\geq 0}|W_A(t)|_H^6\right] = = \mathbb{E}\left[\sup_{t\geq 0}|\int_0^t e^{(t-s)A}\sqrt{Q}dW(s)|^6\right] \leq C \, . \qquad (17)$$

Proof. Denoting by $\|\cdot\|_{HS}$ the Hilbert-Schmidt norm, see, e.g., [11], and by $\mathscr{L}(H)$ the standard operator norm and then exploiting the Burkholder-Davis-Gundy inequality and the boundness of Q, we have

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\geq 0}|W_A(t)|_H^6\right] \\ &= C_m\mathbb{E}\left[\sup_{t\geq 0}|\int_0^t \|e^{(t-s)A}\sqrt{Q}\|_{HS}^2 ds|^3\right] \\ &\leq C\mathbb{E}\left[\sup_{t\geq 0}\int_0^t \|e^{(t-s)A}\|_{\mathscr{L}(H)}^2 Tr(Q)^6 ds\right] \ , \end{aligned}$$

where the last inequality follows from standard properties of a Hilbert-Schmidt operator, see, e.g. [11]. Moreover, by Prop. 1, point (i), we have that

$$\int_0^t \|e^{(t-s)A}\|_{\mathscr{L}(H)}^2 ds \leq \frac{1}{2\omega}\sum_{k=0}^{+\infty}\frac{1}{\lambda_k} \leq C < \infty \, ,$$

where C is a positive constant independent from t, hence, by assumption on Q, we have that $Tr(Q)^6 < \infty$ and the claim follows. □

As previously outlined, we cannot apply standard existence and uniqueness results in our setting since it is characterized by a non-linear term which is not of Lipschitz type, nevertheless such results can be retrieved exploiting an *m-dissipativity* argument, as proved in Prop. 1 (iii).

Proposition 5. *It exists a unique mild solution to the deterministic eq.* (8)

$$\begin{cases} du_0(t) = [Au_0(t) + F(u_0(t))]dt, t \in [0,+\infty) \, , \\ du_0(0) = u^0 \in D(F) \end{cases} , \qquad (18)$$

furthermore we have that

$$|u_0(t)|_H \leq e^{-(\omega-\eta)t}|u^0|_H, \quad t \geq 0 \, , \qquad (19)$$

Proof. The existence and uniqueness of eq. (18) follows from Prop. 1 and by [11, Th. 7.13].

The estimate (19) follows from the Gronwall's lemma applied to

$$\begin{aligned} \frac{d}{dt}|u_0(t)|_H^2 &\leq \langle Au_0(t), u_0(t)\rangle dt + \langle F(u_0(t)), u_0(t)\rangle \\ &\leq -2(\omega-\eta)|u_0(t)|^2 \, . \end{aligned}$$

□

Again applying Prop. 1 we have that an analogous result holds for the stochastic Cauchy problem (9), in particular we have

Proposition 6. *There exists a unique mild solution $u = u(t)$ of* (9)

$$\begin{cases} du(t) = [Au(t) + F(u(t))]dt + \varepsilon\sqrt{Q}dW(t), \\ u(0) = u^0 \in H \end{cases} ; \qquad (20)$$

s.t. $u(t) \in \mathscr{L}^p(\Omega; C([0,T];H))$ and the following holds

$$\mathbb{E}\left[\sup_{t\geq 0}|u(t)|_H^p\right] < +\infty, \qquad (21)$$

for any $p \in [2,\infty)$.

Proof. Existence and uniqueness follow from Prop. 1 and [11, Th. 7.13].

In order to prove estimate 21, let us define $z(t) := u(t) - W_A(t)$, then we have that z solves

$$\begin{cases} z'(t) = Az(t) + F(z(t) + W_A(t) , \\ z(0) = u^0 \end{cases} ;$$

then by Prop. 1 $(i)-(ii)$, we have that

$$\begin{aligned} \frac{d}{dt}|z(t)|_H^{2a} &= 2a\left\langle z', z\right\rangle |z|_H^{2a-2} = \\ &= 2a\left\langle Az + F + W_A, z\right\rangle |z|_H^{2a-2} \\ &\le -2a\omega|z|_H^{2a} + 2a\left\langle F + W_A, z\right\rangle |z|_H^{2a-2} \\ &\le -2a(\omega-\eta)|z|_H^{2a} + 2a\left|F(W_A)\right| |z|_H^{2a-1} \\ &\le -2a(\omega-\eta)|z|_H^{2a} + \frac{2a}{\bar{\xi}}C_a\left|F(W_A)\right| |z|_H^{2a} + \\ &+ C_a 2a\xi|z|^{2a} , \end{aligned}$$

where, exploiting the inclusion of L^p spaces over bounded domains and with no loss of generality, we have set $p := 2a$ and $\bar{\xi} > 0$, s.t.

$$-2a(\omega-\eta) + 2aC_a\bar{\xi} < 0 ,$$

therefore by 4 and Gronwall's lemma, we obtain

$$\begin{aligned} &|z(t)|_H^{2a} \le e^{(-2a(\omega-\eta)+\bar{\xi}C_a 2a)t}|u^0|_H^{2a} + \\ &+ \frac{2aC_a}{\xi}\int_0^t e^{-2a(\omega-\eta)(t-s)}|F(W_A(s))|_H^{2a}ds , \end{aligned}$$

and by eq. (14), see Prop. 1, we have

$$\begin{aligned} &|F(W_A(t))|_H^{2a} \le \\ &\le C_a(1 + |W_A(t)|^3)^{2a} \le C_{a,3}(1 + |W_A(t)|_H^{6a}) , \end{aligned}$$

and thus exploiting Prop. 4, we can write

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\ge 0}\int_0^t e^{-2a(\omega-\eta)(t-s)}|F(W_A(s))|_H^{2a}ds\right] \\ &\le \tilde{C}\mathbb{E}\left[\sup_{t\ge 0}\int_0^t e^{-2a(\omega-\eta)(t-s)}ds + \right. \\ &\left. + C'_a\int_0^t e^{-2a(\omega-\eta)}ds\right] \le \bar{C} \end{aligned} \tag{22}$$

which concludes the proof.

□

Previously obtained results allow us to study the non-linear term F and write its Taylor expansion around the (mild) solution $u_0(t)$ of

$$\begin{cases} du_0(t) = [Au_0(t) + F(u_0(t))]dt, t \in [0, +\infty) , \\ du_0(0) = u^0 \quad u^0 \in D(F) \end{cases} ;$$

in terms of powers of a *small parameter* ε.

3.2 The asymptotic expansion

In this section we follow the approach developed in [1] to present rigorous results concerning the validity of the asymptotic expansion for the reaction-diffusion Van der Pol equation. Let us define for $0 \le \varepsilon < 1$, the function

$$h(t) = \sum_{k=1}^n \varepsilon^k u_k(t) ,$$

where the functions $u_k(t)$ are $H-$valued, $p-$mean integrable continuous stochastic processes on $[0, +\infty)$ for $p \in [2, \infty)$, then we have that the non-linear map F can be written in terms of its Gâteaux derivatives as

$$\begin{aligned} F(u_0(t) + h(t)) &= F(u_0(t)) + \varepsilon\nabla F(u_0(t))[h(t)] + \\ &+ \frac{1}{2}\varepsilon^2\nabla^{(2)}F(u_0(t))[h(t), h(t)] + \dots \\ \dots &+ \frac{1}{n}\varepsilon^n\nabla^{(n)}F(u_0(t))[h(t), h(t), \dots, h(t)] + \\ &+ R^{(n)}(u_0(t), h(t)) , \end{aligned} \tag{23}$$

and, taking into account the multilinearity of the Gâteaux derivative, namely

$$\begin{aligned} &\frac{1}{j!}\nabla^{(j)}F(u_0(t))[\underbrace{h(t), \dots, h(t)}_{j-terms}] = \\ &= \frac{1}{j!}\sum_{k_1+\dots+k_j=j}^{n_j}\varepsilon^j\nabla^{(k_i)}F(u_0(t))[u_{k_1}(t), \dots, u_{k_j}(t)] , \end{aligned}$$

we can rearrange eq. (23) in order to obtain

$$\begin{aligned} &F(u_0(t) + h(t)) = F(u_0(t)) \\ &+ \sum_{k=1}^n \varepsilon^k\nabla F(u_0(t))[u_k(t)] + \\ &+ \sum_{\substack{j_1, j_2 \in \mathbb{N} \\ j_1+j_2=2}}^n \frac{\varepsilon^{j_1+j_2}}{2!}\nabla^{(2)}F(u_0(t))[u_{j_1}(t), u_{j_2}(t)] + \dots \\ &+ \sum_{\substack{j_i \in \mathbb{N} \\ \sum_{i=1}^k j_i = k}}^n \frac{\varepsilon^{j_1+\dots+j_k}}{k!}\nabla^{(k)}F(u_0(t))[u_{j_1}(t), ..., u_{j_k}(t)] \\ &+ \dots + \frac{\varepsilon^n}{n!}\nabla^{(n)}F(u_0(t))[u_1(t), \dots, u_1(t)] + \\ &+ R_1^{(n)}(u_0(t), h(t), \varepsilon) . \end{aligned} \tag{24}$$

where we have introduced

$$\begin{aligned} &R_1^{(n)}(u_0(t), h(t), \varepsilon) = \\ &= \sum_{j=2}^n \sum_{i_1+\dots+i_j=n+1}^{n_j} \varepsilon^{i_1+\dots+i_j} \times \\ &\times \frac{1}{j!}\nabla^{(j)}F(u_0(t))[u_{i_1}(t), \dots, u_{i_j}(t)] + \\ &+ R^{(n)}(u_0(t), h(t)) , \end{aligned} \tag{25}$$

with the remainder $|R^{(n)}(u_0(t),h(t))|_H \leq C(u_0,n)|h|_H^n$, see, e.g., [18]. We underline that eq. (24) can be also rewritten as follows

$$\begin{aligned} &F(u_0(t)+h(t)) = F(u_0(t))+ \\ &+\sum_{j=2}^{n} \sum_{i_1+\cdots+i_j=n+1}^{n_j} \varepsilon^{i_1+\cdots+i_j} times \\ &\times \frac{1}{j!}\nabla^{(j)}F(u_0(t))[u_{i_1}(t),\ldots,u_{i_j}(t)]+ \\ &+R^{(n)}(u_0(t),h(t)), \end{aligned} \tag{26}$$

see, e.g. [1].

Lemma 7. *Let be $R_1^{(n)}(u_0(t),h(t),\varepsilon)$ as in eq.* (25)*, then for all $p \in [2,\infty)$ there exists a constant $C > 0$, depending on $u_0,\ldots,u_n,\nabla F^{(1)},\ldots,\nabla F^{(n)},p$, s.t.*

$$\mathbb{E}\left[\sup_{t\in[0,+\infty)} |R_1(u_0(t),h(t),\varepsilon)|_H^p\right]^{\frac{1}{p}} \leq C\varepsilon^{n+1},$$

for all $0 \leq \varepsilon \leq 1$.

Proof. See, e.g., [1, lemma 4.1]. □

To go forward in the formal expansion, we rewrite $u(t)$ as:

$$u(t) = u_0(t)+\varepsilon u_1(t)+\cdots+\varepsilon^n u_n(t)+R_n(t,\varepsilon), \tag{27}$$

where the processes $u_i(t), i = 1\ldots,n$ can be found by the Taylor expansion of F around $u_0(t)$, and *matching terms* in the equation for u.

Given the stochastic processes $v_0(t),\ldots,v_n(t)$ let

$$\begin{aligned} &\Phi_k(t)[v_0(t),\ldots,v_n(t)] := \\ &= \sum_{j=1}^{k} \sum_{\substack{i_1,\ldots,i_j\in\{1,\ldots,n\} \\ \sum_{l=1}^{j} i_l = k}} \nabla^{(j)}F(v_0(t))[v_{i_1}(t),\ldots,v_{i_j}(t)], \end{aligned} \tag{28}$$

then, exploiting Prop. $1-(iv)$, we have that $\nabla^{(k)} = 0$, for $k \geq 4$, hence, if we take $n = 3$, we have the following

Proposition 8. *Let us consider Φ_k as in eq.* (28) *and set $n = 3$, then*

$$\begin{aligned} &\Phi_2(t)[v_0(t),\ldots,v_3(t)] = \\ &=\nabla^{(1)}F(v_0(t))[v_2(t)]+\frac{1}{2}\nabla^{(2)}F(v_0(t))[v_1(t),v_1(t)], \\ &\Phi_3(t)[v_0(t),\ldots,v_3(t)] = \\ &= \nabla^{(1)}F(v_0(t))[v_3(t)]+\nabla^{(2)}F(v_0(t))[v_1(t),v_2(t)] \\ &+\nabla^{(3)}F(v_0(t))[v_1(t),v_1(t),v_1(t)], \end{aligned}$$

Proof. It is a straightforward application of eq. (28) together with Prop. 1. □

Previous result implies that, taking $n = 3$, the terms $u_1(t),\ldots,u_3(t)$ defined in (27), are solution to

$$\begin{aligned} &du_1(t) = [Au_1(t)+\nabla F(u_0(t))[u_1(t)]]dt+\sqrt{Q}dW(t), \\ &u_1(0) = 0, \\ &du_i(t) = [Au_2(t)+\nabla F(u_0(t))[u_i(t)]]dt+\Phi_i(t)dt, \\ &u_i(0) = 0, \quad i = 2,3, \end{aligned}$$

with $\Phi_j(t) := \Phi_j(t)[u_0(t),\ldots,u_{j-1}(t)]$ as in Prop. 8 without the v_j-th term, namely

$$\begin{aligned} &\Phi_2(t) = \frac{1}{2}\nabla^{(2)}F(v_0(t))[v_1(t),v_1(t)], \\ &\Phi_3(t) = \nabla^{(2)}F(v_0(t))[v_1(t),v_2(t)]+ \\ &\quad +\nabla^{(3)}F(v_0(t))[v_1(t),v_1(t),v_1(t)]. \end{aligned} \tag{29}$$

Proposition 9. *We have that*

$$\begin{aligned} &\nabla F(u_0(t))[u_k(t)] = \\ &= \left(-3u_0^2(t)+2u_0(t)(\xi_1+1)-\xi_1\right)u_k(t), \end{aligned}$$

for any $k = 1,2,3,\ldots$. Furthermore we have

$$\begin{aligned} &\Phi_2(t) = (-3u_0(t)+\xi_1+1)\,u_1^2(t), \\ &\Phi_3(t) = (-6u_0(t)+ \\ &\quad +2(\xi_1+1))\,u_1(t)u_2(t)-6u_0(t)u_1^3(t). \end{aligned}$$

Proof. From Prop. 1 (iv) we immediately have that

$$\begin{aligned} &\nabla F(u_0(t))[u_k(t)] = \\ &= \left(-3u_0^2(t)+2u_0(t)(\xi_1+1)-\xi_1\right)u_k(t),, \\ &\nabla^{(2)}F(v_0(t))[v_i(t),v_j(t)] = \\ &= (-6u_0(t)+2(\xi_1+1))\,u_i(t)u_j(t), \\ &\nabla^{(3)}F(v_0(t))[v_i(t),v_j(t),v_k(t)] = \\ &= 6u_0(t)u_i(t)u_j(t)u_k(t), \end{aligned} \quad, \tag{30}$$

substituting now (30) into eq. (29) we prove the claim. □

Let us notice that while $u_1(t)$ is the solution of a linear stochastic differential equation, the processes u_2,u_3 are solutions of differential equations with a stochastic non-homogeneus term in the following sense

Definition 10. *Let $i = 1,2$, then a predictable H-valued stochastic process $u_i = u_i(t)$,$t \geq 0$ is a solution of problem:*

$$\begin{cases} du_i(t) = [Au_i(t)+\nabla F(u^0(t))[u_i(t)]]dt+\Phi_i(t)dt, \\ u_i(0) = 0 \end{cases};$$

if for almost every $\omega \in \Omega$ it satisfies

$$\begin{aligned} u_i(t) = &\int_0^t e^{(t-s)A}\nabla F(u_0(s))[u_i(s)]ds+ \\ &+\int_0^t \Phi_i(s)ds,\ t \geq 0,\ i = 2,3, \end{aligned}$$

with Φ_i as in eq. (29)*.*

The following result will be used to estimate the norm of Φ_i in H by means of the norms of the Gâteaux derivatives of F and the norms of (H-valued stochastic process) $v_j(t)$, $j = 1,\ldots,i-1$.

Lemma 11. *Let $v_0(t), v_1(t), v_2(t)$ be H-valued stochastic processes. Then we have the following inequalities*

$$\begin{aligned}
&|\Phi_2(t)\,[v_0(t), v_1(t)]|_H \leq \\
&\leq C_2 |v_0(t)|_H 8(2 + |v_1(t)|_H|_H)\,,\\
&|\Phi_3(t)\,[v_0(t), v_1(t), v_2(t)]|_H \leq \\
&\leq C_3 |v_0(t)|_H 27(3 + |v_1(t)|_H^2 + |v_2(t)|_H^2),
\end{aligned}$$

C_2 depends on $\|\nabla^{(2)}F\|_{\mathscr{L}(H;\mathscr{L}(H^2;H))}$ and C_3 depends on $\|\nabla^{(i)}F\|_{\mathscr{L}(H;\mathscr{L}(H^2;H))}$, for $i = 2, 3$.

Proof. The rigorous proof can be found in [1, Lemma 4.3], the claim follows taking into account applying Lemma 4.3 together with eq. (29) to the present setting. ☐

4 Main results

Exploiting the preparatory results obtained so far, in what follows we shall show our main contributions

Proposition 12. *For any $p \geq 2$, the equation*

$$\begin{cases} du_1(t) = [Au_1(t) + \nabla F(u_0(t))[u_1(t)]]\,dt + \sqrt{Q}dW(t)\,, \\ u_1(0) = 0, t \in [0, +\infty) \end{cases}; \tag{31}$$

has a unique mild solution which satisfies the following estimate:

$$\mathbb{E}\left[\left(\sup_{t\in[0,+\infty)} |u_1(t)|_H^p\right)^{1/p}\right] < +\infty\,, \tag{32}$$

Proof. The proof is a slight modification of that given in [11], we gain the case $p = 2$ by using the $m-$dissipativity of F proved in Prop. 1. Estimate (32) is analogous to the one given in Prop. 6. ☐

Theorem 13. *Let $i = 2, 3$ and let u_1 be the solution of the problem* (31). *Then there exists a unique mild solution $u_i(t)$, $t \in [0, +\infty)$, of*

$$\begin{cases} du_i(t) =[Au_i(t) + \nabla F(u^0(t))[u_i(t)]]dt + \Phi_i(t)dt\,, \\ u_i(0) = 0 \end{cases}; \tag{33}$$

which $\forall p \in [2, \infty)$ satisfies

$$\mathbb{E}\left[\left(\sup_{t\in[0,+\infty)} |u_i(t)|_H^p\right)^{\frac{1}{p}}\right] < +\infty. \tag{34}$$

Proof. The proof proceed iteratively and it is based on classical fixed point theorem for contraction, where the contraction is given by

$$\Gamma(y(t)) := \int_0^t e^{(t-s)A}\nabla F(u_0(t))[y(t)]ds + \\ + \int_0^t e^{(t-s)A}\Phi_k(s)ds\,,$$

on the space $\mathscr{L}^p(\Omega; C([0,T]); H)$. Proceeding as in Prop. 6 we have the existence and uniqueness of a mild solution for eq. (33), see, e.g. [1, Th. 5.2].

The estimate (34) is obtained exploiting smoothness properties of the operator ∇F, then, see also Prop. 6, we have

$$\frac{d}{dt}|u_k(t)|_H^{2a} \leq -2aC_a^1|u_k(t)|_H^{2a} + C_a^2|\Phi_k(t)|_H^{2a}\,, \tag{35}$$

with $C_a^1 = \omega - \gamma(1 + |u_0|)$.

By estimate (32) on $u_1(t)$ and lemma 11 we have that

$$\mathbb{E}\left[\left(\sup_{t\in[0,+\infty)} |\Phi_2(t)|_H^{2a}\right)^{\frac{1}{2a}}\right] < C_a' < +\infty\,.$$

Taking the expectation in inequality (35) and by Gronwall's lemma we obtain

$$\mathbb{E}\left[\left(\sup_{t\in[0,T]} |u_2(t)|_H^{2a}\right)^{\frac{1}{2a}}\right] \leq C_a' e^{-2aC_a^1 T} < C_a\,,$$

where C_a is a positive constant independent of T. The case $i = 3$ is treated analogously, then exploiting the bound on u_2, see also [1, Th. 5.2], we have the thesis. ☐

Collecting previously obtained results we are now in position to prove the following main theorem

Theorem 14. *The solution $u(t)$ of* (7) *can be expanded in the following form*

$$u(t) = u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \varepsilon^3 u_3(t) + R_3(t, \varepsilon),$$

where u_0 solves

$$\begin{cases} du_0(t) = [Au_0(t) + F(u_0(t))]dt, t \in [0, +\infty)\,, \\ du_0(0) = u^0 \in D(F) \end{cases}; \tag{36}$$

u_1 is the solution of

$$\begin{cases} du_1(t) &= [Au_1(t) + \nabla F(u_0(t))u_1(t)]dt + \sqrt{Q}dW(t)\,, \\ u_1(0) &= 0 \end{cases}; \tag{37}$$

while u_i, $i = 2, 3$, is the solution of

$$\begin{cases} du_i(t) &= [Au_i(t) + \nabla F(u_0(t))u_i(t)]dt + \Phi_i(t)dt\,, \\ u_i(0) &= 0 \end{cases}; \tag{ACP(k)}$$

with

$$\begin{aligned}
&\nabla F(u_0(t))[u_k(t)] = \\
&= \left(-3u_0^2(t) + 2u_0(t)(\xi_1 + 1) - \xi_1\right)u_k(t)\,, \quad k = 1, 2, 3\,,\\
&\Phi_2(t) = \left(-3u_0(t) + \xi_1 + 1\right)u_1^2(t),\\
&\Phi_3(t) = \left(-6u_0(t) + 2(\xi_1 + 1)\right)u_1(t)u_2(t) - 6u_0(t)u_1^3(t)\,.
\end{aligned} \tag{38}$$

Furthermore we have that $R_3(t,\varepsilon)$ is given by

$$\begin{aligned} R_3(t,\varepsilon) &= u(t) - u_0(t) - \sum_{i=1}^{3} \varepsilon^i u_i(t) \\ &= \int_0^t e^{(t-s)A} \left(F(u(s)) - F(u_0(s)) + \right. \\ &\left. - \sum_{i=1}^{3} \varepsilon^i \nabla F(u_0(t))[u_i(s)] - \sum_{i=2,3} \Phi_i(s) \right) ds\,, \end{aligned} \tag{39}$$

and it verifies the following inequality

$$\mathbb{E}\left[\left(\sup_{t\in[0,+\infty)} |R_3(t,\varepsilon)|_H^p \right)^{1/p} \right] \le C(p)\varepsilon^4,$$

where $C(p)$ is a constant depending on p

Proof. It follows from [1, 5.3] applying Prop. 1 and Prop. 8. In particular we have that the mild solutions for u, u_0, u_1, u_k, $k = 2,3$ are respectively given by

$$\begin{aligned} u(t) &= e^{tA}u^0 + \int_0^t e^{(t-s)A} F(u(s))ds + \varepsilon W_A(t)\,, \\ u_0(t) &= e^{tA}u^0 + \int_0^t e^{(t-s)A} F(u_0(s))ds\,, \\ u_1(t) &= \int_0^t e^{(t-s)A} F(u_0(s))[u_1(s)]ds + \varepsilon W_A(t)\,, \\ u_k(t) &= \int_0^t e^{(t-s)A} F(u_0(s))[u_k(s)]ds + \\ &\quad + \int_0^t e^{(t-s)A} \Phi_k(s)ds\,, \quad k = 2,3\,, \end{aligned}$$

then the particular forms of $F(u_0(s))[u_k(s)]$ and of Φ_i given in eq. (38), follow from Prop. 9.

Eventually from eq. (39) and exploiting Prop. 1 (i), we have that

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\in[0,+\infty)} |R_3(t,\varepsilon)|_H^p \right] \le \\ &\le \mathbb{E} \sup_{t\in[0,+\infty)} \left| \int_0^t e^{(t-s)A} R_1^{(n)}(u_0(s);h(s),\varepsilon)ds \right|_H^p \\ &\le \mathbb{E} \sup_{t\in[0,+\infty)} \int_0^t \left\| e^{(t-s)A} \right\|_{\mathscr{L}(H,H)}^p \left| R_1^{(n)}(u_0;h,\varepsilon) \right|_H^p ds \\ &\le C_{n,p}\varepsilon^{p(n+1)} \mathbb{E} \sup_{t\in[0,+\infty)} \left| R_1^{(n)}(u_0(s);h(s),\varepsilon) \right|_H^p\,, \end{aligned}$$

and the claim follows by Lemma 7.

□

5 Conclusion

In the present work we have studied the infinite dimensional stochastic reaction-diffusion equation of the Van der Pol type, with external forcing, perturbed by a Brownian motion whose intensity is governed by a small parameter ε. For such a model we have provided an explicit expression for the related expansion in power series with respect to a small parameter ε. It is worth to mention that analogous results can be investigated when a more general noise has been considered, as in the case where the driving noise is a general Lévy type process, see, e.g. [3]. In particular we would like to underline that the case of a general Lévy process is highly interesting from a financial point of view, where this kind of equation is exploited in order to study energy markets, namely market which are characterized by high uncertainty with empirical evidence showing the occurrence of jumps and systematic seasonal effects, see, e.g. [8, 19, 20]. A different possible application concerns the neurostochastics framework, when the reference equation is as in the *FitzHugh-Nagumo* model, see, e.g., [6].

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